ISRAEL JOURNAL OF MATHEMATICS, Vol. 32, Nos. 2-3, 1979

NEW AXIOMATIZATIONS FOR LOGICS WITH GENERALIZED QUANTIFIERS†

BY
VICTOR HARNIK AND MICHAEL MAKKAI

ABSTRACT

We give a complete axiomatization for admissible fragments of $L_{\omega_1\omega}(Q)$. This axiomatization implies syntactically Gregory's characterization of $L_{\omega_1\omega}$ sentences with no uncountable models ([5]). This is then extended to stationary logic. To obtain these results, we employ Ressayre's methods ([16], [17]) augmented with an application of game sentences. In section 4 we prove a result emphasizing the naturalness of Gregory's result.

§0. Introduction.

The following elegant characterization result was proven by J. Gregory ([5]).

THEOREM 0.1. *For a countable admissible set $\mathscr{A}$ and a $\Sigma_{\mathscr{A}}$-theory $T \subset L_{\mathscr{A}}$, the following three conditions are equivalent*:

(i) *T has no uncountable model*;

(ii) *T has no pair of distinct countable models $\mathfrak{A}_0, \mathfrak{A}_1$ such that $\mathfrak{A}_0 \prec_{L_{\mathscr{A}}} \mathfrak{A}_1$*;

(iii) *$T \vdash \theta$ for some $\theta \in \mathscr{C}_{L_{\mathscr{A}}}(v = v)$.*

Here $\mathscr{C}_{L_{\mathscr{A}}}(v = v)$ is a set of $L_{\mathscr{A}}$ formulas described in the sequel; each element of it "says" in a natural way that the universe (i.e. the set of elements satisfying $v = v$) is countable.

To describe $\mathscr{C}(v = v)$ (we will usually suppress the reference to $L_{\mathscr{A}}$) we define by induction $\mathscr{C}(\phi)$ for every $L_{\mathscr{A}}$ formula $\phi(v, \boldsymbol{x})$ (more precisely, the binary relation $\psi \in \mathscr{C}(\phi)$ will be defined by induction on the complexity of (ϕ, ψ)). The members of $\mathscr{C}(\phi)$ will be $L_{\mathscr{A}}$ formulas $\theta(\boldsymbol{x}, \boldsymbol{y})$ ($\boldsymbol{y}$ a, possibly empty, sequence of additional free variables) such that for all $\mathfrak{A}$, $\mathfrak{A} \models \theta[\boldsymbol{a}, \boldsymbol{b}]$ implies that $\phi(\mathfrak{A}, \boldsymbol{a}) =$

† Research supported by a grant from the National Research Council of Canada.

Received June 17, 1977

$\{c \in A : \mathfrak{A} \models \phi[c, \boldsymbol{a}]\}$ is countable. The clauses of Definition 0.2 below reflect the following simple principles concerning countable sets: (I) any singleton is countable; (II) any subset of a countable set is countable; (III) any countable union of countable sets is countable.

DEFINITION 0.2. $\mathscr{C}(\phi)$, for $\phi = \phi(v, \boldsymbol{x})$, is defined by the following clauses subject to the restriction that all formulas are in $L_{\mathscr{A}}$.

(1) (singleton clause) t (= true) $\in \mathscr{C}(v = x)$;

(2) (*subset clause*) *if* $\theta' \in \mathscr{C}(\phi')$ then $\forall v(\phi \to \phi') \wedge \theta' \in \mathscr{C}(\phi)$;

(3) (union clause) if $\theta_i \in \mathscr{C}(\phi_i)$, $i \in I$, then $\bigwedge_{i \in I} \theta_i \in \mathscr{C}(\bigvee_{i \in I} \phi_i)$.

(p.c.) (parameter clause) if $\theta \in \mathscr{C}(\phi)$ and y is a variable which is not free in ϕ then $\exists y \theta \in \mathscr{C}(\phi)$.

As said, (1)–(3) reflect principles (I)–(III) while (p.c.) reflects a trivial logical principle.

Returning to Theorem 0.1, the implications (iii)$\to$(ii)$\to$(i) are obvious whereas (i)$\to$(iii) is Gregory's achievement; while his main concern was with the equivalences between (i) and (ii), we are interested in the equivalence between (i) and (iii). The equivalence (i)$\leftrightarrow$(iii) can be viewed as a *partial* completeness theorem for the logic $L_{\mathscr{A}}(Q)$ (Q meaning "there are uncountably many"). Indeed, let $\mathscr{S}_0$ be the deductive system obtained from first order logic by considering as new additional axioms all statements of the form $\theta \to \neg Qv(v = v)$ with $\theta \in \mathscr{C}(v = v)$. Then (i)$\leftrightarrow$(iii) says that for a $\Sigma_{\mathscr{A}}$ theory T, $T \models \neg Qv(v = v)$ iff $\neg Qv(v = v)$ is provable from T in $\mathscr{S}_0$; in fact, $\neg Qv(v = v)$ has a simply structured proof consisting of a first order derivation of a $\theta \in \mathscr{C}(v = v)$ followed by one new axiom and modus ponens. Our main aim was to generalize this result. We exhibit a deductive system $\mathscr{S}$ having the following properties:

(a) $\mathscr{S}$ is complete for $L_{\mathscr{A}}(Q)$ (Theorem 1.2);

(b) $\mathscr{S}_0$ is a subsystem of $\mathscr{S}$;

(c) any $\mathscr{S}$-derivation of $\neg Qv(v = v)$ from a theory $T \subset L_{\mathscr{A}}$ as above can be easily converted to a simply structured proof of the kind described above (Corollary 1.6).

The equivalence of (i) and (ii) in Theorem 0.1 was proven subsequently by a new, simpler and more general method by Ressayre [16], [17], notably using $\Sigma_{\mathscr{A}}$-saturated structures, or, as we are going to call them here, $\mathscr{A}$-recursively saturated structures. (For $\mathscr{A} = HF$ these were defined independently by Barwise and Schlipf [3] who also noticed the applicability of recursively saturated structures to the related Vaught two-cardinal theorem.) By the same approach

Ressayre obtained a substitute for $\mathscr{C}(v = v)$ and a related complete deductive system for $L_{\mathscr{A}}(Q)$. However, Ressayre's axioms are not very intuitive and, in particular, conditions (b) and (c) above are not met by his system.

We adopt in this paper Ressayre's method and augment it by an application of game sentences. These turn out to be particularly useful as they suggest the axioms of the system (cf. proof of Lemma 3.4).

A feature of our system shared by Ressayre's but not by Keisler's well known system ([9]) is that for deriving a sentence ϕ one does not need to use statements starting with Q other than those which occur as subformulas of ϕ. This is crucial for purpose (c) above since the only Q-statement occurring in any $\mathscr{S}$-derivation of $\neg Qv(v = v)$ from $T(\subset L_{\mathscr{A}})$ is $Qv(v = v)$.

In section 1 we present our system, spell out its relation to Theorem 0.1, and discuss some connections with Keisler's system.

In section 2 we summarize the needed background connected to game sentences and $\mathscr{A}$-recursively saturated structures.

In section 3 we prove our completeness theorem.

In section 4 we prove a result emphasizing the naturalness of Gregory's result.

In section 5 we use our technique to study a variant of stationary logic ([18], [2]). Again, the game sentences suggest the axioms of a complete system.

§1. A completeness theorem for $L_{\mathscr{A}}(Q)$

$L_{\omega_1\omega}(Q)$ is the logic obtained from $L_{\omega_1\omega}$ by allowing the formation of $Qx\phi(x)$ for any formula $\phi(x)$; the meaning of $Qx\phi(x)$ is "for uncountably many $x, \phi(x)$". $L_{\mathscr{A}}(Q)$ is $L_{\omega_1\omega}(Q) \cap \mathscr{A}$. Taking up a definition of Ressayre [16], a *fragment* F is a countable set of $L_{\omega_1\omega}(Q)$ formulas closed under taking subformulas, substitutions of free variables, finitary Boolean combinations and first order quantification but not necessarily under Q-quantification (thus, it may happen that $Qu\psi(u) \in F$ but $Qv\psi(v) \notin F$!). F is called admissible if for some countable admissible $\mathscr{A}$, F is a $\Delta_{\mathscr{A}}$ subset of $L_{\mathscr{A}}(Q)$ and it is closed under $\mathscr{A}$-finite disjunctions and conjunctions.

We adopt the approach of Ressayre and regard Q-quantification (with respect to a variable v) as a syntactical operation which associates to any formula $\phi(v, x_0, \cdots, x_{n-1})$ a new n-ary atomic formula $Qv\phi(v, x_0, \cdots, x_{n-1})$. An *F-structure* will be one in which all atomic formulas of F (including those of the form $Qv\phi$) are interpreted. By a *weak model* of a set of sentences in F we mean an F-structure satisfying the given sentences. An F-structure $\mathfrak{A}$ is called *standard* if for all $Qv\phi \in F$, $\boldsymbol{a} \in A$, $\mathfrak{A} \models Qv\phi[v, \boldsymbol{a}]$ iff for uncountably many $b \in A$, $\mathfrak{A} \models \phi[b, \boldsymbol{a}]$.

Notice that if $\mathscr{A}$ is admissible then $L_{\mathscr{A}}$ itself is an example of an admissible fragment. We now extend Definition 0.2 and define $\mathscr{C}_F(\phi)$ for any admissible fragment F and any $\phi = \phi(v, \boldsymbol{x}) \in F$. Whenever the context is clear enough we shall allow ourselves to suppress F from the notation.

Definition 1.1. $\mathscr{C}_F(\phi)$ is defined by the following clauses subject to the restriction that all formulas involved belong to F.

Clauses (1)–(3) and (p.c.) of Definition 0.2, and

(4) (Q-clause) if $\theta(y) \in \mathscr{C}(\phi)$ then

$$\forall y \theta(y) \in \mathscr{C}(\exists y(\neg Qu\psi(u) \wedge \psi(y) \wedge \phi(v, y)))$$

(θ, ϕ, ψ may have free variables other than v and y).

For $F = L_{\mathscr{A}}$, clause (4) is void, so Definition 0.2 is, indeed, a particular case of Definition 1.1. The intuitive meaning of $\mathscr{C}_F(\phi)$ is similar to that of $\mathscr{C}_{L_{\mathscr{A}}}(\phi)$. The principle behind the new clause is, again, that a countable union of countable sets is countable; the Q-clause must be added because the Q-quantifier introduces one more way of saying that a definable set is a countable union of definable sets; indeed, the formula $\exists y\,(\neg Qu\psi(u) \wedge \psi(y) \wedge \phi(v, y))$ defines the union of the sets defined by the instances of $\phi(v, y)$ for the (countably many) values of y satisfying $\psi(y)$; if $\theta(y)$ "says" that $\phi(v, y)$ defines a countable set then $\forall y \theta(y)$ "will say" that the existential formula above defines a countable set.

Let $T_0(F)$ be the set of universal closures of all formulas of the form $\theta \to \neg Qv\phi$, where $Qv\phi \in F$ and $\theta \in \mathscr{C}_F(\phi)$. Notice that $T_0(F)$ is $\mathscr{A}$-recursive (because so is the binary relation $\psi \in \mathscr{C}(\phi)$). Obviously, all elements of $T_0(F)$ are valid in all standard F-structures. We can now describe a deductive system $\mathscr{S}(F)$:

The axioms of $\mathscr{S}(F)$ are the usual first order axioms of F (cf. e.g. [10], page 15) as well as all elements of $T_0(F)$. (*Remark*: when defining syntactical notions such as "the variable v is free for the term t in ϕ" we consider Q as a usual quantifier.)

The rules of $\mathscr{S}(F)$ are the usual first order rules, i.e. Modus Ponens, Generalization and Conjunction (from $\{\phi \to \psi : \psi \in \Psi\}$ infer $\phi \to \wedge \Psi$, whenever Ψ is $\mathscr{A}$-finite).

We denote by $\vdash_{\mathscr{S}(F)}$ derivability in $\mathscr{S}(F)$.

Obviously, $\mathscr{S}(F)$ is a sound deductive system. The main result of this paper is the completeness of $\mathscr{S}(F)$:

Theorem 1.2. *If F is admissible then $\mathscr{S}(F)$ is complete, i.e., whenever $T \cup \{\phi\} \subset F$ and T is $\Sigma_{\mathscr{A}}$ then $T \models \phi$ implies that $T \vdash_{\mathscr{S}(F)} \phi$.*

By the completeness theorem for first order admissible fragments (e.g. [10]), Theorem 1.2 is equivalent to the following first order model theoretic statement:

THEOREM 1.3. *Let F be admissible and $T \subset F$ be a $\Sigma_{\mathscr{A}}$-theory. If $T \cup T_0(F)$ has a weak model then it has a standard model.*

We prove Theorem 1.3 in §3. We devote the rest of this section to present a few applications of our completeness theorem.

As is always the case, in order to be able to use our deductive system efficiently, we have to derive a few theorems of $\mathscr{S}(F)$. For this purpose we introduce the notation "$\theta \succ \phi$ (in F)" to mean that $\theta, \phi \in F$ and there is some $\theta' \in \mathscr{C}_F(\phi)$ such that $\theta \vdash \theta'$ (here and below "$\vdash$" means first order derivability and $\theta \vdash \theta'$ is synonymous with $\vdash \theta \to \theta'$). In general, we shall omit mentioning "(in F)". Usually, we shall write "$\succ \phi$" for "$\mathrm{t} \succ \phi$". Obviously, if $\theta \succ \phi$ and $Qv\phi \in F$ then $\theta \to \neg Qv\phi$ is a theorem of $\mathscr{S}(F)$. The reader should convince himself that whenever we claim below that $\theta \succ \phi$ for some particular θ and ϕ we can actually exhibit a $\theta' \in \mathscr{C}(\phi)$ and a derivation of it from θ. We need the following:

LEMMA 1.4. (i) *The relation $\theta \succ \phi$ (in F) satisfies clauses* (1)–(4) *and* (p.c.) *from the Definition* 1.1 *of $\theta \in \mathscr{C}_F(\phi)$;*

(ii) *if $\theta \succ \phi$ and $\phi' \vdash \phi$ then $\theta \succ \phi'$; if $\theta \succ \phi$ and $\theta' \vdash \theta$ then $\theta' \succ \phi$;*

(iii) *for all θ, $\theta \succ \mathrm{f}$ (the false sentence);*

(iv) *if $\theta(y) \succ \phi(v, y)$ then $\theta(z) \succ \phi(v, z)$ provided that y is free for z in $\phi(v, y)$;*

(v) *if $\theta \succ \phi$ then $\succ \theta \wedge \phi$;*

(vi) $\neg Qu\psi(u) \succ \psi(v)$.

PROOF. (i) is trivial except for the fact that the proof of clause (3) requires a Σ-collection; (ii) and (iii) are immediate; for (iv), first prove by induction on the definition of $\mathscr{C}(\phi)$ that if $\theta(y) \in \mathscr{C}(\phi(v, y))$ then $\theta(z) \in \mathscr{C}(\phi(v, z))$; (v) is due to Gregory (for the case $F = L_{\mathscr{A}}$). One first shows by induction on the definition of $\mathscr{C}(\phi)$ that $\theta \in \mathscr{C}(\phi)$ implies $\succ \theta \wedge \phi$. We exemplify two steps of the induction:

Clause (4). Let $\theta(y) \in \mathscr{C}(\phi(v, y))$ and assume (inductive assumption) that $\succ \theta(y) \wedge \phi(v, y)$; by (ii), $\mathrm{t} \succ \forall y \theta(y) \wedge \phi(v, y)$; by (i),

$$\forall y\, \mathrm{t} \succ \exists y(\neg Qu\psi(u) \wedge \psi(y) \wedge \forall y \theta(y) \wedge \phi(v, y));$$

finally, by (ii) again, $\succ \forall y \theta(y) \wedge \exists y(\neg Qu\psi(u) \wedge \psi(y) \wedge \phi(v, y))$;

Parameter clause (*p.c.*). Let $\theta(y) \in \mathscr{C}(\phi)$, y not free in ϕ and assume that $\succ \theta(y) \wedge \phi$; by (i),

$$\forall v(\exists y\theta(y)\wedge\phi\rightarrow\theta(y)\wedge\phi)\succ\exists y\theta(y)\wedge\phi;$$

by (i) again, applying (p.c.), $\exists y\forall v(\exists y\theta(y)\wedge\phi\rightarrow\theta(y)\wedge\phi)\succ\exists y\theta(y)\wedge\phi$; the formula on the left side is valid, hence $\succ\exists y\theta(y)\wedge\phi$; (vi): by applying clause (4) to $v=y$ as $\phi(v,y)$ and t as $\theta(y)\in\mathscr{C}(\phi(v,y))$ one gets, by (ii), $\succ\neg Qu\psi(u)\wedge\psi(v)$; as $\neg Qu\psi(u)\vdash\forall v(\psi(v)\rightarrow\neg Qu\psi(u)\wedge\psi(v))$, the conclusion follows.

In preparation for our first application, we need one more fact about $\succ$.

PROPOSITION 1.5 (Gregory for $F=L_{\mathscr{A}}$). *If $\theta_i\succ\phi$ for all $i\in I$, then $\bigvee_{i\in I}\theta_i\succ\phi$.*

PROOF. By Lemma 1.4 (v), $\succ\theta_i\wedge\phi$ for all $i\in I$, hence, by Lemma 1.4 (i), $\succ\bigvee_{i\in I}(\theta_i\wedge\phi)$. By Lemma 1.4 (i) again, $\forall v(\phi\rightarrow\bigvee_{i\in I}(\theta_i\wedge\phi))\succ\phi$; the conclusion now follows by the fact that the formula on the left is implied by $\bigvee_{i\in I}\theta_i$.

Gergory's result that (i) and (iii) in Theorem 0.1 are equivalent is a particular case (corresponding to $F=L_{\mathscr{A}}$) of the following corollary of our completeness theorem.

COROLLARY 1.6. *Assume that F is admissible, ϕ has only v free, $T\cup\{\phi(v)\}\subset F$ and T is $\Sigma_{\mathscr{A}}$. The following are equivalent:*

(i) $T\vDash\neg Qv\phi(v)$,

(iii) $T\vdash_{\mathscr{S}(F)}\theta$ *for some* $\theta\in\mathscr{C}_F(\phi)$.

PROOF. (iii) $\Rightarrow$ (i) is trivial.

(i) $\Rightarrow$ (iii). If $Qv\phi\in F$ then this is trivial by Theorems 1.2 and 1.4 (vi). So assume $Qv\phi\notin F$. Let F_1 be the smallest admissible fragment containing F and $Qv\phi$. Thus, for an $\mathscr{A}$-finite formula, $\psi\in F_1$ iff any Q-subformula (i.e. a subformula starting with Q) of ψ is either $Qv\phi$ or else belongs to F.

By Theorem 1.2, (i) implies that $T\vdash_{\mathscr{S}(F_1)}\neg Qv\phi$. Hence there are $\mathscr{A}$-finite sets $A_0,A_1\subset T_0(F_1)$ such that: A_0 is a set of axioms of the form $\theta\rightarrow\neg Qu\psi$ where $Qu\psi\in F$, $\theta\in\mathscr{C}_{F_1}(\psi)$; $A_1=\{\theta_i\rightarrow\neg Qv\phi\}_{i\in I}$ where $\theta_i\in\mathscr{C}_{F_1}(\phi)$ and $T\cup A_0\vdash\wedge A_1\rightarrow\neg Qv\phi$. ('$\vdash$' denotes first order derivability.) $Qv\phi$, which has several occurrences in the derivation above, is considered, syntactically speaking, an atomic sentence. The notion of first order derivability is preserved if we substitute each of these occurrences by, say, t. Thus, denoting by χ^* the result of substituting t for $Qv\phi$ in χ and letting $S^*=\{\chi^*:\chi\in S\}$ for any set of formulas S, we have

$$T^*\cup A_0^*\vdash\wedge\{\theta_i^*\rightarrow\mathrm{f}\}\rightarrow\mathrm{f}.$$

By first order propositional calculus this yields

$$T^* \cup A_0^* \vdash \bigvee_{i \in I} \theta_i^*.$$

A couple of observations will finish the proof.

First, if $\chi \in F$ then $\chi^* = \chi$, hence $T^* = T$ and a typical member of A_0^* is $\theta^* \rightarrow \neg Qu\psi$ where $\theta \in \mathscr{C}_{F_1}(\psi)$.

Second, if $\theta \in \mathscr{C}_{F_1}(\chi)$ then $\theta^* \succ \chi^*$ (in F) as seen by a simple induction on the definition of $\mathscr{C}_{F_1}(\chi)$. (We indicate one step of the induction: let $\theta(y) \in \mathscr{C}(\chi(v, y))$ and assume that $\theta^*(y) \in \mathscr{C}(\chi^*)$; we want to show that

$$(\forall y\theta)^* \succ [\exists y(\neg Qu\psi(u) \wedge \psi(y) \wedge \chi)]^*;$$

Case 1. $Qu\psi \in F$; then we have to show $\forall y\theta^* \succ \exists y(\neg Qu\psi(u) \wedge \psi(y) \wedge \chi^*)$ which is trivial by Lemma 1.4 (i);

Case 2. $Qu\psi = Qv\phi$; then we have to show that $\forall y\theta^* \succ \exists y(\mathrm{f} \wedge \phi(y) \wedge \chi^*)$ which is trivial again by Lemma 1.4 (iii) and (ii).)

Thus, the typical member of A_0^* is $\theta^* \rightarrow \neg Qv\psi$ with $\theta^* \succ \psi$ (in F), hence it is a theorem of $\mathscr{S}(F)$! Also, $\theta_i^* \succ \phi$ for all $i \in I$. By Proposition 1.5, $\bigvee_{i \in I} \theta_i^* \succ \phi$. Summing up, we reach the desired conclusion: $T \vdash_{\mathscr{S}(F)} \bigvee_{i \in I} \theta_i^* \succ \phi$.

REMARK. For F-structures $\mathfrak{A}_0$, $\mathfrak{A}_1$ we say that $\mathfrak{A}_1$ is a *good* extension of $\mathfrak{A}_0$ if $\mathfrak{A}_0 <_F \mathfrak{A}_1$ and, whenever $Qu\psi(u, \mathbf{y}) \in F$, $\mathbf{a} \in A_0$, $\mathfrak{A}_0 \models \neg Qu\psi[u, \mathbf{a}]$ implies that

$$\psi(\mathfrak{A}_1, \mathbf{a}) = \psi(\mathfrak{A}_0, \mathbf{a})(= \{b : b \in A_0 \text{ and } \mathfrak{A}_0 \models \psi[b, \mathbf{a}]\}).$$

Then (i) and (iii) in Corollary 1.6 are equivalent to

(ii) there is no pair $\mathfrak{A}_0, \mathfrak{A}_1$ of weak models of $T \cup T_0(F)$ s.t. $\mathfrak{A}_1$ is a good extension of $\mathfrak{A}_0$ and $\phi(\mathfrak{A}_1) \neq \phi(\mathfrak{A}_0)$.

The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are easily seen to hold. We get, thus, a full generalization of Gregory's result, Theorem 0.1.

By the same method we can get various other characterization results as corollaries of the Completeness Theorem 1.2. We illustrate one more example. Let $\phi(y, z)$, $\psi(v)$ be $L_{\mathscr{A}}$-formulas; given any structure $\mathfrak{A}$, we say that *ϕ is $\aleph_1$-like with respect to ψ in $\mathfrak{A}$* if $\psi(\mathfrak{A})$ is uncountable while $\phi(\mathfrak{A}, a) = \{b : \mathfrak{A} \models \phi[b, a]\}$ is countable for all $a \in A$ (the best known instance of this situation is that of an $\aleph_1$-like ordering w.r.t. $v = v$). Fix $\phi(y, z)$. For any $\chi(v) \in L_{\mathscr{A}}$ define $\mathscr{D}(\chi)$ by clauses (1)–(3) and (p.c.) of Definition 0.2 as well as the clause:

(4) if $\theta(y) \in \mathscr{D}(\chi(v, y))$ then $\forall y\theta(y) \in \mathscr{D}(\exists y(\phi(y, z) \wedge \chi(v, y)))$. It is easily seen that every $\theta \in \mathscr{D}(\chi)$ "says" that ϕ is not $\aleph_1$-like with respect to χ.

COROLLARY 1.7. *Let* $T\cup\{\phi,\psi\}\subset L_{\mathscr{A}}$ *with* T Σ *over* $\mathscr{A}$. *The following are equivalent*:

(a) T *has no model in which* ϕ *is* $\aleph_1$*-like* w.r.t. ψ;

(b) $T\vdash\theta$ *for some* $\theta\in\mathscr{D}(\psi)$.

PROOF. (Sketch). (a) together with Theorem 1.2 implies that for a suitable F,

$$T\vdash_{\mathscr{S}(F)}\forall z\neg Qy\phi(y,z)\rightarrow\neg Qv\psi(v).$$

Substitute, in the derivation, f and t for $Qy\phi(y,z)$ and $Qv\psi(v)$, respectively. An analysis which is analogous to the one in the previous proof yields condition (b).

REMARK. For the case of $\aleph_1$-like orderings, a simpler syntactic characterization exists and is due to Keisler [8] (the idea is that the axioms of order together with an additional "regularity axiom" insure that any first order model can be transformed into a *weak model* of Keisler's axioms [9] by interpreting $Qv\phi(v)$ as "for unboundedly many $x,\phi(x)$").

The best known complete deductive system for $L(Q)$ is that of Keisler ([9]). It is not complete for admissible fragments F in the sense that the proof of a validity in F may require axioms which go beyond F. On the other hand, Keisler's system is complete for any (not necessarily admissible) fragment closed under Q (cf. *loc. cit.* for a precise definition). The axioms of Keisler's system, slightly modified, are grouped in the following four schemes which bear a remarkable resemblance to the clauses of our Definition 1.1:

K1. $\neg Qv(v=y)$,

K2. $\forall v(\phi\rightarrow\phi')\wedge\neg Qv\phi'\rightarrow\neg Qv\phi$,

K3. $\bigwedge_{i\in I}\neg Qv\phi_i\rightarrow\neg Qv(\bigvee_{i\in I}\phi_i)$,

K4. $\forall y\neg Qv\phi(v,y)\wedge\neg Qu\psi(u)\rightarrow\neg Qv\exists y(\psi(y)\wedge\phi(v,y))$

(one scheme of Keisler's, namely $Qu\psi(u)\rightarrow Qv\psi(v)$, was made superfluous by the fact that the variables u and v in K4 were allowed to be distinct).

Due to this resemblance, our axioms are easily derived from Keisler's and while checking this, we come to the conclusion that for proving a validity $\phi\in F$ (F admissible) in Keisler's system we need not apply Q more than *once* on any particular F-formula. To be more precise, let F' be the smallest admissible fragment containing F and all formulas of the form $Qu\psi$ with $\psi\in F$. Let $T\vdash'\phi$ denote that ϕ has a Keisler derivation from T involving only F'-formulas.

COROLLARY 1.8. *Let* F,T,ϕ *be as in Theorem* 1.2. *If* $T\vDash\phi$ *then* $T\vdash'\phi$.

PROOF. We show by induction on the definition of $\mathscr{C}_F(\phi)$ that for all

$\theta \in \mathscr{C}(\phi)$, we have $\vdash' \theta \to \neg Qv\phi$. The only step which is not absolutely trivial is the one connected to clause (4) (this is due to the slight dissimilarity between this clause and K4). Assuming that (1) $\vdash' \theta(y) \to \neg Qv\phi(v, y)$ for some $\theta(y) \in \mathscr{C}(\phi)$, we show

(4) $\vdash' \forall y\theta \to \neg Qv\exists y(\neg Qu\psi(u) \wedge \psi(y) \wedge \phi(v, y))$

as follows: from (1) we get by K4 and first order logic:

(2) $\vdash' \forall y\theta(y) \to \neg(\neg Qu\psi(u) \wedge Qv\exists y(\psi(y) \wedge \phi(v, y)))$.

It is an easy exercise to show, using K3, that if $\gamma, \delta(v) \in F$ and v is not free in γ then $\vdash' Qv(\gamma \wedge \delta(v)) \to \gamma \wedge Qv\delta(v)$. Hence we get from (2):

(3) $\vdash' \forall y\theta(y) \to \neg Qv(\neg Qu\psi(u) \wedge \exists y(\psi(y) \wedge \phi(v, y)))$.

One more application of K3 yields (4).

We conclude this section with a few remarks and open questions.

(1) A close examination of Keisler's proof in [9] reveals that it yields Corollary 1.8 for the case of $F \subset L_{\omega\omega}$. However, the case of an infinitary F does not seem to follow from [9] since that proof uses strong omission which involves unrestricted applications of Q. Neither does Bruce's [4] seem to imply Corollary 1.8.

QUESTION. Is Corollary 1.8 true for fragments F which are not admissible? (One should assume that F contains $\bigvee_{i \in I} Qv\phi_i$ whenever it contains $Qv\bigvee_{i \in I} \phi_i$.)

(2) One could replace K4 by

K4'. $\forall y \neg Qv\phi(v, y) \to \neg Qv\exists y(\neg Qu\psi(u) \wedge \psi(y) \wedge \phi(v, y))$

in complete analogy with clause (4) of Definition 1.1. This would make the proof of Corollary 1.8 absolutely trivial. As far as F'-instances go, K4' is equivalent to K4 modulo $\vdash'$ derivations (as seen by the argument in Corollary 1.8).

A natural thought would be to replace clause (4) by the more elegant

(4') if $\theta(y) \in \mathscr{C}(\phi(v, y))$ then

$$\forall y\theta(y) \wedge \neg Qu\psi(u) \in \mathscr{C}(\exists y(\psi(y) \wedge \phi(v, y)))$$

which is analogous to K4. Let $\mathscr{C}'_F$ and $\mathscr{S}'(F)$ be defined with this modified clause. Unfortunately, $\mathscr{S}'(F)$ does not seem to be complete; we believe that the valid

$$\phi = \neg Qv\exists y(\neg QuP(u) \wedge P(y) \wedge \neg QwR(w, y) \wedge R(v, y))$$

is not provable in $\mathscr{S}'(F)$ (F being the fragment generated by ϕ).

(3) There are two problems which arise naturally in the present context. The first is to find a primitive recursive procedure which converts any Keisler

derivation of a validity ϕ into a derivation from $\mathcal{S}(F)$, F being the least admissible fragment containing ϕ. If this is done then we get a proof of Gregory's Theorem 0.1 via Keisler's completeness theorem.

The second problem is the converse one: find a primitive recursive procedure which would transform any $\mathcal{S}(F)$-derivation of a validity ϕ into a Keisler derivation inside the Q-closed (not necessarily admissible) fragment generated by ϕ. Maybe a solution to this would elucidate the question we asked in Remark (1).

§2. Preliminaries on game sentences and recursively saturated structures

For the definition of the notion of conjunctive game sentence Φ and its $L_{\omega_1\omega}$ approximations Φ_α, we refer the reader to [12], [6], [7]. We shall mainly use the dual notion of disjunctive game (or: co-Vaught) sentence. With the notation $\boldsymbol{k}_n = \langle k_0, \cdots, k_n \rangle$ and similarly for $\boldsymbol{j}_n$, $\boldsymbol{x}_n$, etc., a typical disjunctive game-sentence is:

$$(*) \qquad \Psi = \left(\exists x_0 \bigvee_{k_0 \in K_0} \forall x_1 \bigwedge_{j_0 \in J_0} \cdots \exists x_{2n} \bigwedge_{k_n \in K_n} \forall x_{2n+1} \bigvee_{j_n \in J_n} \cdots \right) \bigvee_{n < \omega} \psi^{\boldsymbol{k}_n, \boldsymbol{j}_n}(\boldsymbol{x}_{2n+1}),$$

where the sets $K = \{\langle n, k \rangle : k \in K_n, n < \omega\}$ and $J = \{\langle n, j \rangle : j \in J_n, n < \omega\}$ are called the *index-sets* of Ψ. The $L_{\omega_1\omega}$ approximations Ψ_α are defined as the obvious duals of the approximations of conjunctive game sentences, cf. *loc. cit.*; at the appropriate places in this paper the reader will see how the approximations are formed.

By the determinateness of closed games the negation of any conjunctive game sentence Φ is equivalent to a disjunctive game-sentence which we denote by $\Phi\neg$.

Given $K' \subset K$, we denote by $\Psi_{K'}$ the *stronger* sentence obtained from Ψ by replacing each K_n by K'_n ($= \{k : \langle n, k \rangle \in K'\}$) in $(*)$. The approximations of $\Psi_{K'}$ are denoted by $\Psi_{K',\alpha}$.

In the sequel, $\mathcal{A}$ denotes a fixed but arbitrary countable admissible set. Since $\omega \in \mathcal{A}$ is not assumed, our results and proofs are valid for finitary logic.

Ψ is called a $\Sigma_{\mathcal{A}}$-disjunctive game sentence if K, J and the function $\langle \boldsymbol{k}_n, \boldsymbol{j}_n \rangle \mapsto \psi^{\boldsymbol{k}_n, \boldsymbol{j}_n}$ are $\Sigma_{\mathcal{A}}$ and if each J_n is $\mathcal{A}$-finite for $n < \omega$. (The dual notion of $\Sigma_{\mathcal{A}}$ conjunctive game sentence is studied in [6].)

The notion of $\Sigma_{\mathcal{A}}$-saturated structure was introduced and used in Ressayre [17]; for the finitary case cf. also Barwise and Schlipf [3]. We are going to use the name '$\mathcal{A}$-recursively saturated structure' (abbreviated $\mathcal{A}$-r.s.) instead. For the sake of completeness, we reproduce the definition here. Consider formulas of the form $\Phi(\boldsymbol{y}) = \exists x \bigvee_{j\in J} \bigwedge_{k\in K_j} \phi^{jk}(x,\boldsymbol{y})$ where $\phi^{jk} \in L_{\mathscr{A}}$, J is $\mathscr{A}$-finite and the set $K = \{\langle j,k\rangle: k \in K_j\}$ as well as the function $\langle i,j\rangle \to \phi^{jk}$ are $\Sigma_{\mathscr{A}}$; for $K' \subset K$, denote by $\Phi_{K'}$ the formula obtained from Φ after replacing K_j by $K'_j = \{k: \langle j,k\rangle \in K'\}$. A structure $\mathfrak{A}$ is $\mathscr{A}$-r.s. if for each $\Phi(\boldsymbol{y})$ as above and for each $\boldsymbol{a} \in \mathscr{A}$, $\mathfrak{A} \models \phi[\boldsymbol{a}]$ iff $\mathfrak{A} \models \Phi_{K'}[\boldsymbol{a}]$ for each $\mathscr{A}$-finite $K' \subset K$. In the following two theorems, we summarize some properties of this notion. For details, cf. [17] or [12].

THEOREM 2.1. (i) *If $\mathfrak{A}$ is an $\mathscr{A}$-r.s. structure, then for any $\Sigma_{\mathscr{A}}$-disjunctive game sentence Ψ, $\mathfrak{A} \models \Psi$ if and only if $\mathfrak{A} \models \Psi_{K',\alpha}$ for some $\mathscr{A}$-finite $K' \subset K$ and for some $\alpha <$* Ord$(\mathscr{A})$.

(ii) *Every consistent $\Sigma_{\mathscr{A}}$-theory of $L_{\mathscr{A}}$ has an $\mathscr{A}$-r.s. model.*

(iii) *The union of an $L_{\mathscr{A}}$-elementary chain of $\mathscr{A}$-r.s. structures is $\mathscr{A}$-r.s.*

A very important property of $\mathscr{A}$-r.s. structures is what was called by Ressayre strong-relation universality (cf. [17] and [12]). We will need two consequences of this property. To state the first of these, we introduce the following terminology. Let $L, L' \subset \mathscr{A}$ be languages Δ over $\mathscr{A}$, $L \subset L'$, $U \in L' - L$ a unary predicate and $T' \subset L'_{\mathscr{A}}$ a $\Sigma_{\mathscr{A}}$-theory. A *T'-pair* is a pair $(\mathfrak{A},\mathfrak{B})$ of L-structures such that $\mathfrak{A} \prec_{L_{\mathscr{A}}} \mathfrak{B}$ and $\mathfrak{B}$ has an expansion $\mathfrak{B}' \models T'$ such that $A = |\mathfrak{A}| = U^{\mathfrak{B}'}$.

THEOREM 2.2. ([17]). (i) *If $\mathfrak{A}$ is $\mathscr{A}$-r.s. and there is a T'-pair $(\mathfrak{A}',\mathfrak{B}')$ with $\mathfrak{A}' \equiv_{L_{\mathscr{A}}} \mathfrak{A}$, then there is an $\mathscr{A}$-r.s. $\mathfrak{B}$ such that $(\mathfrak{A},\mathfrak{B})$ is a T'-pair.*

(ii) *If $\mathfrak{A}$ is $\mathscr{A}$-r.s. then there is an admissible set $\mathscr{B} \supset \mathscr{A}$ with* Ord$(\mathscr{B}) =$ Ord$(\mathscr{A})$ *and a $\mathscr{B}$-finite structure $\mathfrak{B}$ which is isomorphic to $\mathfrak{A}$.*

Finally, in this paper we use game sentences with *restricted* (or relativized) quantifiers. More specifically, together with (*) we are given two $\Sigma_{\mathscr{A}}$-sequences $\langle \lambda_n(\boldsymbol{x}_{2n})\rangle_{n<\omega}$ and $\langle \rho_n(\boldsymbol{x}_{2n+1})\rangle_{n<\omega}$ of $L_{\mathscr{A}}$-formulas and we are told that the quantifiers $\exists x_{2n}$ and $\forall x_{2n+1}$ in the prefix of Ψ are *restricted* (or relativized) to $\lambda_n(\boldsymbol{x}_{2n})$ and $\rho_n(\boldsymbol{x}_{2n+1})$ respectively. We may indicate such restrictions by writing $\exists_{\lambda_n} x_{2n}$ or $\forall_{\rho_n} x_{2n+1}$. The meaning of such a restriction is, of course, that e.g. the range of the variable x_{2n} is restricted to the set of elements x_{2n} satisfying $\lambda_n(\boldsymbol{x}_{2n-1}, x_{2n})$.

These restrictions affect the definition of the approximations. For example, the definition of $\Psi^{k_n,j_n}_{\alpha+1}$ becomes:

$$\Psi^{k_n,j_n}_{\alpha+1}(\boldsymbol{x}_{2n+1}) = \exists x_{2n+2}(\lambda_{n+1}(\boldsymbol{x}_{2n+2}) \wedge \bigvee_{k_{n+1}\in K_{n+1}} \forall x_{2n+3}(\rho_{n+1}(\boldsymbol{x}_{2n+3}) \to \bigwedge_{j_{n+1}\in J_{n+1}} \Psi^{k_{n+1},j_{n+1}}_{\alpha}(\boldsymbol{x}_{2n+3}))).$$

With this notion of approximation, Theorem 2.1 (i) remains true.

§3. Proof of the Completeness Theorem

We have to show that if F is admissible and $T \subset F$ is a $\Sigma_{\mathscr{A}}$-theory such that $T \cup T_0(F)$ has a (weak) model, then T has a standard model.

In this section "$<$" will denote the elementary substructure relationship with respect to F. We recall from §1 that, for countable F-structures $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{B}$ is a good extension of $\mathfrak{A}$ if $\mathfrak{A} < \mathfrak{B}$ and for every $Qu\psi(u, \boldsymbol{y}) \in F$, $\boldsymbol{a} \in A$, if $\mathfrak{A} \models \neg Qu\psi[u, \boldsymbol{a}]$ then $\psi(\mathfrak{A}, \boldsymbol{a}) = \psi(\mathfrak{B}, \boldsymbol{a})$.

Keisler's method ([9]) for building a standard model relies on the following observation. Assume that we have an F-elementary chain of countable structures $\{\mathfrak{A}_\alpha\}_{\alpha<\omega_1}$ s.t.

(1) $\mathfrak{A}_0 \models T$,

(2) $\mathfrak{A}_{\alpha+1}$ is a good extension of $\mathfrak{A}_\alpha$,

(3) for each $Qv\phi(v, \boldsymbol{x}) \in F$ and $\boldsymbol{a} \in A_\alpha$, if $\mathfrak{A}_\alpha \models Qv\phi[v, \boldsymbol{a}]$ then there is a cofinal set $X \subset \omega_1 - \alpha$ such that for all $\beta \in X$, there is an element $c \in A_{\beta+1} - A_\beta$ with $\mathfrak{A}_{\beta+1} \models \phi[c, \boldsymbol{a}]$. Under these assumptions, $\mathfrak{A}_{\omega_1} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$ is a standard model of T.

Assuming that $T \cup T_0(F)$ has a model we follow Ressayre ([16], [17]) and build a chain as above with each $\mathfrak{A}_\alpha$ an $\mathscr{A}$-r.s. model of $T \cup T_0(F)$. The construction is by induction on α; by Theorem 2.1 (ii), there is an $\mathscr{A}$-r.s. $\mathfrak{A}_0 \models T \cup T_0(F)$; for a limit λ, $\mathfrak{A}_\lambda = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha$ is $\mathscr{A}$-r.s. by Theorem 2.1 (iii), provided that each $\mathfrak{A}_\alpha, \alpha < \lambda$ is such; to construct $\mathfrak{A}_{\alpha+1}$, apply the following lemma for $\mathfrak{A} = \mathfrak{A}_\alpha$.

LEMMA 3.1. *If $\mathfrak{A}$ is a countable $\mathscr{A}$-r.s. model of $T_0(F)$, $Qv\phi(v, \boldsymbol{x}) \in F$ and $\mathfrak{A} \models Qv\,[v, \boldsymbol{a}]$ for some $\boldsymbol{a} \in A$, then there exists an $\mathscr{A}$-r.s. structure $\mathfrak{A}_1$ which is a good extension of $\mathfrak{A}$ and such that $\mathfrak{A}_1 \models \phi[c, a]$ for some element $c \in A_1 - A$.*

We now come to the core of our proof. By applying Theorem 2.2 (i) to a theory T' "coding" the type of extension appearing in Lemma 3.1, we see that it is sufficient to show:

LEMMA 3.2. *If $\mathfrak{A}$ is a countable $\mathscr{A}$-r.s. model of $T_0(F)$ then for each $Qv\phi(v, \boldsymbol{x}) \in F$ and $\boldsymbol{a} \in A$, if $\mathfrak{A} \models Qv\phi[v, \boldsymbol{a}]$ then there is a substructure $\mathfrak{A}_0 < \mathfrak{A}$ such that $\mathfrak{A}$ is a good extension of $\mathfrak{A}_0$, $\boldsymbol{a} \in A_0$ and $\mathfrak{A} \models \phi[c, \boldsymbol{a}]$ for some $c \in A - A_0$.*

PROOF. We first outline our argument. We will exhibit a conjunctive game formula $\Phi(v, \boldsymbol{x})$ expressing the following statement:

> "There is a substructure $\mathfrak{A}_0 < \mathfrak{A}$ such that $\mathfrak{A}$ is a good extension of $\mathfrak{A}_0$, $\boldsymbol{x} \in A_0$ and $v \in A - A_0$".

We have to show that for each $Qv\phi(v, \boldsymbol{x}) \in F$,

$$\mathfrak{A} \models \forall \boldsymbol{x}(Qv\phi(v, \boldsymbol{x}) \rightarrow \exists v(\phi(v, \boldsymbol{x}) \wedge \Phi(v, \boldsymbol{x}))).$$

If this is not true then for some $Qv\phi \in F$ and some $\boldsymbol{a} \in A$,

(1) $\mathfrak{A} \models Qv\phi[v, \boldsymbol{a}]$

and $\mathfrak{A} \models \forall v(\phi(v, \boldsymbol{a}) \rightarrow \Psi(v, \boldsymbol{a}))$ where Ψ is a disjunctive game formula equivalent to $\neg \Phi$ (Ψ is obtained by "pushing the negation inside" in $\neg \Phi$). If so, then Theorem 2.1 (i) (applied to the disjunctive game formula $\forall_\phi v \Psi(v, \boldsymbol{x})$ which is, trivially, equivalent to $\forall v(\phi \rightarrow \Psi)$) yields some $\mathscr{A}$-finite K' and α such that

$$\mathfrak{A} \models \forall v(\phi(v, \boldsymbol{a}) \rightarrow \Psi_{K',\alpha}(v, \boldsymbol{a})).$$

We shall show that for every K' and α there is a *valid* $\theta_{K',\alpha}(\boldsymbol{x}) \in \mathscr{C}_F(\Psi_{K',\alpha}(v, \boldsymbol{x}))$. Thus, $\theta = \forall v(\phi \rightarrow \Psi_{K',\alpha}) \wedge \theta_{K',\alpha} \in \mathscr{C}_F(\phi)$ and

(2) $\mathfrak{A} \models \theta[\boldsymbol{a}]$.

Since $\forall \boldsymbol{x}(\theta(\boldsymbol{x}) \rightarrow \neg Qv\phi(v, \boldsymbol{x})) \in T_0(F)$, (1) and (2) contradict the fact that $\mathfrak{A} \models T_0(F)$. This concludes the outline of the proof. The details which follow from here to the end of this section will complete the proof of Lemma 3.2, and thus the proof of Theorem 1.3 as well.

We now write down Φ explicitly. Assume that $y_0, y_1, \cdots$ is a sequence of variables distinct from v and $\boldsymbol{x}$. Let

$$\Phi(v, \boldsymbol{x}) = \bigwedge_{\delta_0} \exists y_0 \bigwedge_{\gamma_0} \forall y_1 \bigwedge_{\delta_1} \exists y_2 \cdots \left(\bigwedge_{n<\omega} (\delta_n \wedge v \neq y_{2n} \wedge v \neq y_{2n+1}) \right)$$

where:

(i) δ_n ranges over all F-formulas of the form $\exists y \chi(\boldsymbol{x}, \boldsymbol{y}_{2n-1}, y) \rightarrow \chi(\boldsymbol{x}, \boldsymbol{y}_{2n})$;

(ii) γ_n ranges over

$$\{\gamma(\boldsymbol{x}, \boldsymbol{y}_{2n}, u): u \text{ any variable}, Qu\gamma \in F\} \cup \left\{ \bigvee_{i \leq 2n} u = y_i : u \text{ any variable} \right\}.$$

(iii) $\forall y_{2n+1}$ is restricted to ν_{γ_n} where ν_{γ_n} is defined as $\neg Qu\gamma_n(\boldsymbol{x}, \boldsymbol{y}_{2n}, u) \wedge \gamma_n(\boldsymbol{x}, \boldsymbol{y}_{2n+1})$ if $Qu\gamma_n \in F$ and as $\bigvee_{i \leq 2n} y_{2n+1} = y_i$ if γ_n is $\bigvee_{i \leq 2n} u = y_i$ (this exceptional clause in the definition of ν_{γ_n} is somewhat arbitrary and needed for a technical purpose; for reasons which will become clear in a moment, we call it the "escape clause").

Φ has, indeed, the intended meaning:

Claim 3.3. *For any countable F-structure $\mathfrak{A}$, $\mathfrak{A} \models \Phi[c, \boldsymbol{a}]$ iff there is $\mathfrak{A}_0 < \mathfrak{A}$ such that $\mathfrak{A}$ is a good extension of $\mathfrak{A}_0$, $\boldsymbol{a} \in A_0$ and $c \in A - A_0$.*

PROOF. We show the "only if" direction (actually, only this direction is needed for the proof of Lemma 3.2).

Assume that $\mathfrak{A} \models \Phi[c, \boldsymbol{a}]$. This means that player $\exists$ has a winning strategy in the game suggested by the prefix of Φ. In a typical play of this game $\exists$ and its opponent $\forall$ take turns, $\exists$ picking the elements $b_{2n} \in A$ (corresponding to y_{2n}) and $\forall$ picking formulas δ_n, γ_n and elements $b_{2n+1} \in A$ (corresponding to y_{2n+1}). Let $\exists$ play its winning strategy against a play of $\forall$ which satisfies the following conditions:

(a) the sequence $\boldsymbol{\delta}_\omega = \langle \delta_n : n < \omega \rangle$ enumerates all formulas of the form $\exists y \chi(\boldsymbol{x}, y_{2n-1}, y) \to \chi(\boldsymbol{x}, y_{2n})$;

(b) for γ_n, $\forall$ chooses a formula $\gamma_n = \gamma(\boldsymbol{x}, \boldsymbol{y}_{2n}, u)$ such that the set $\nu_\gamma(\boldsymbol{a}, \boldsymbol{b}_{2n}, \mathfrak{A}) = \{d : \mathfrak{A} \models \nu_\gamma[\boldsymbol{a}, \boldsymbol{b}_{2n}, d]\}$ is non-empty (if the "luck is bad" and there is no such γ with $Qu\gamma \in F$ then $\forall$ always can use the escape clause and pick γ_n to be $\bigvee_{i \leq n} u = y_i$); moreover, for any $Qu\gamma(\boldsymbol{x}, \boldsymbol{y}, u) \in F$ and $\boldsymbol{b}$ in $\boldsymbol{b}_\omega$ with a non-empty $\nu_\gamma(\boldsymbol{a}, \boldsymbol{b}, \mathfrak{A})$ there should be infinitely many $n < \omega$ with $\gamma(\boldsymbol{a}, \boldsymbol{b}, u) = \gamma_n(\boldsymbol{a}, \boldsymbol{b}_{2n}, u)$;

(c) for all n, $\forall$ chooses the element b_{2n+1} such that $\mathfrak{A} \models \nu_{\gamma_n}[\boldsymbol{a}, \boldsymbol{b}_{2n}, b_{2n+1}]$; by the above, this is always possible. Moreover, for every $Qu\gamma \in F$ and $\boldsymbol{b}$ with a non-empty $\nu_\gamma[\boldsymbol{a}, \boldsymbol{b}, \mathfrak{A}]$, the set

$$\{b_{2n+1} : \gamma(\boldsymbol{a}, \boldsymbol{b}, u) = \gamma_n(\boldsymbol{a}, \boldsymbol{b}_{2n}, u)\}$$

should exhaust all of $\gamma(\boldsymbol{a}, \boldsymbol{b}, \mathfrak{A})$.

Once the game has been played in this way, $A_0 = \boldsymbol{a} \cup \{b_n : n < \omega\}$ is the universe of an $\mathfrak{A}_0$ as desired. Indeed, condition (a) together with the fact that $\boldsymbol{b}_\omega$, $\boldsymbol{\delta}_\omega$, $\boldsymbol{a}$, c satisfy the matrix formulas of Φ insures that $\mathfrak{A}_0 < \mathfrak{A}$ and $c \in A - A_0$ while conditions (b) and (c) insure that $\mathfrak{A}$ is a good extension of $\mathfrak{A}_0$. This completes the proof of Claim 3.3.

We now consider Ψ, the logical equivalent of $\neg \Phi$. Suppressing, for notational simplicity, any mention of $\boldsymbol{x}$, $\Psi = \Psi(v)$ is

$$\bigvee_{\delta_0} \forall y_0 \bigvee_{\gamma_0} \exists y_1 \cdots \left(\bigvee_{n < \omega} \delta_n(\boldsymbol{y}_{2n+1}) \to v = y_{2n} \vee v = y_{2n+1} \right).$$

The definition of the approximations of $\Psi(v)$ depends on a partition of the prefix into blocks of equal (finite) length. We choose the following definition for $\Psi_\alpha^{\delta_n, \gamma_n}(v, \boldsymbol{y}_{2n+1})$:

(1) $\Psi_0^{\boldsymbol{\delta}_n, \boldsymbol{\gamma}_n} = \bigvee_{i \leq n} (\delta_i \to v = y_{2i} \vee v = y_{2i+1})$,

(2) $\Psi_{\alpha+1}^{\boldsymbol{\delta}_n, \boldsymbol{\gamma}_n} = \bigvee_{\delta_{n+1}} \forall y_{2n+2} \bigvee_{\gamma_{n+1}} \exists y_{2n+3} (\nu_{\gamma_{n+1}} \wedge \Psi_\alpha^{\boldsymbol{\delta}_{n+1}, \boldsymbol{\gamma}_{n+1}})$,

(3) $\Psi_\lambda^{\boldsymbol{\delta}_n, \boldsymbol{\gamma}_n} = \bigvee_{\alpha < \lambda} \Psi_\alpha^{\boldsymbol{\delta}_n, \boldsymbol{\gamma}_n}$ for a limit λ.

For $n = -1$, this definition yields the approximations $\Psi_\alpha(v) = \Psi_\alpha^{\varnothing,\varnothing}(v)$.

According to the outline we set for our proof of Lemma 3.2 we have now to exhibit a valid $\theta \in \mathscr{C}_F(\Psi_{K'},\alpha(v))$ for each suitable $\mathscr{A}$-finite K' and α. The existence of such a θ follows from a more general statement which is easily proven by induction on the above definition of the approximations:

CLAIM 3.4. *For every $\mathscr{A}$-finite K' and α and for every $n < \omega$, $\boldsymbol{\delta}_n$ and $\boldsymbol{\gamma}_n$,*

$$\bigwedge_{i \leq n} \delta_i \succ \Psi_{k',\alpha}^{\boldsymbol{\delta}_n,\boldsymbol{\gamma}_n}.$$

PROOF. For a given K', the proof goes by induction on α. Once K' is fixed, let us suppress it from the notation. We exhibit in a primitive recursive way formulas $\theta_\alpha^{\boldsymbol{\delta}_n,\boldsymbol{\gamma}_n}$ for which it will be clear that

$$\bigwedge_{i \leq n} \delta_i \vdash \theta_\alpha^{\boldsymbol{\delta}_n,\boldsymbol{\gamma}_n} \succ \Psi_\alpha^{\boldsymbol{\delta}_n,\boldsymbol{\gamma}_n}.$$

The induction is straightforward provided one uses Lemma 1.4 (i) as well as:

PROPOSITION 3.5. (i) $\forall v(\phi \rightarrow \bigvee_{i \leq m} v = y_i) \succ \phi$;
(ii) *if* $\theta(y) \succ \phi(v, y)$ *then* $\forall y \theta(y) \succ \exists y(\bigvee_{i \leq m} y = y_i \wedge \phi(v, y))$;
(iii) *if* $\theta(y) \succ \phi(v, y)$ *then* $\exists y \theta(y) \succ \forall y \phi(v, y)$.

PROOF OF PROPOSITION 3.5. (i) follows easily, by Lemma 1.4 (i) using clauses (1), (3), (2) of Definition 1.1; (ii) by Lemma 1.4 (iv), $\theta(y_i) \succ \phi(v, y_i)$ and the result follows by the fact that $\forall y \theta(y) \vdash \bigwedge_{i \leq m} \theta(y_i) \succ \bigvee_{i \leq m} \phi(v, y_i)$, this last formula being equivalent to $\exists y(\bigvee_{i \leq m} y = y_i \wedge \phi(v, y))$.

(iii) is equally trivial, using clause (3) of Definition 1.1.

CONCLUSION OF THE PROOF OF CLAIM. 3.4 (and hence of the completeness theorem).

For $\alpha = 0$,

$$\bigwedge_{i \leq n} \delta_i \vdash \forall v\Big(\Psi_0^{\boldsymbol{\delta}_n,\boldsymbol{\gamma}_n} \rightarrow \bigvee_{i \leq 2n+1} v = y_i\Big)$$

and, by Proposition 3.5 (i), we can choose this last formula as $\theta_0^{\boldsymbol{\delta}_n,\boldsymbol{\gamma}_n}$.

For the successor step *from α to $\alpha + 1$*, simplifying notation we suppress $\boldsymbol{\delta}_n$, $\boldsymbol{\gamma}_n$, v, $\boldsymbol{y}_n$ and write δ, γ, y, z for $\delta_{n+1}, \gamma_{n+1}, y_{2n+2}, y_{2n+3}$, respectively, and we have

$$\Psi_{\alpha+1} = \bigvee_{\delta} \forall y \bigvee_{\gamma} \exists z (\nu_\gamma(y, z) \wedge \Psi_\alpha^{\delta,\gamma}(y, z))$$

and by the induction assumption for all δ, γ (satisfying the restriction imposed by K') there is $\theta^{\delta,\gamma}$ s.t.

$$\bigwedge_{i \leqq n} \delta_i \wedge \delta \vdash \theta^{\delta,\gamma} \succ \Psi_\alpha^{\delta,\gamma}.$$

As $z(= y_{2n+3})$ does not occur in $(\bigwedge_{i \leqq n} \delta_i) \wedge \delta$, we have

$$\bigwedge_{i \leqq n} \delta_i \wedge \delta \vdash \forall z \theta^{\delta,\gamma} \quad \text{and} \quad \forall z \theta^{\delta,\gamma} \succ \exists z (\nu_\gamma(y,z) \wedge \Psi_\alpha^{\delta,\gamma}(y,z))$$

either by Lemma 1.4 (i) and clause (4) of Definition 1.1 or by Proposition 3.5 (ii) according to whether ν_γ is $\neg Qu\gamma(y,u) \wedge \gamma(y,z)$ or $\bigvee_{i \leqq 2n+2} z = y_i$. Now by applying, successively, clause (3) of Definition 1.1, Proposition 3.5 (iii) and again clause (3) of Definition 1.1 we have

$$\theta = \bigwedge_\delta \exists y \bigwedge_\gamma \forall z \theta^{\delta,\gamma} \succ \Psi_{\alpha+1}.$$

It is obvious that θ is implied by $\bigwedge_\delta \exists z (\bigwedge_{i \leqq n} \delta_i \wedge \delta)$. As z does not appear in δ_i, $i \leqq n$ and $\exists z \delta$ is valid, we have that, in fact,

$$\bigwedge_{i \leqq n} \delta_i \vdash \theta.$$

The successor case of the induction is, thus, complete.

We leave the limit case to the reader.

The proof of Theorem 1.3 is now complete.

REMARK. The natural inductive definition of $\theta_\alpha^{\delta_n,\gamma_n}$ suggests the clauses of a possible definition of $\mathscr{C}_F(\phi)$. Our actual Definition 1.1 is the result of eliminating obvious redundancies from these clauses. At any rate, this was the way in which we arrived at the crucial clause (4) in Definition 1.1.

§4. The admissible hull and categoricity of the Scott-sentence of a structure

We give a characterization of those countable structures $\mathfrak{A}$ whose Scott sentence SS($\mathfrak{A}$) is categorical: $\mathfrak{B} \models$ SS($\mathfrak{A}$) implies that $\mathfrak{B}$ is countable (equivalently, that $\mathfrak{B} \simeq \mathfrak{A}$); to put it in other words, $\mathfrak{A}$ is the model of a categorical $L_{\omega_1\omega}$-sentence. The characterization is given in terms of HYP$_{\mathfrak{A}}$, the *admissible hull* of $\mathfrak{A}$. For this notion, cf. [1], or also [15]. Roughly speaking, our result will state that $\mathfrak{A}$ is as above iff it is countable in virtue of principles (1)–(3) (cf. the Introduction) applied "inside" HYP$_{\mathfrak{A}}$. For simplicity, let us assume that $\mathfrak{A}$ has a *finite* similarity type. One considers V_A, the universe of all sets based on the set of urelements taken to be the domain A of the structure $\mathfrak{A}$. HYP$_{\mathfrak{A}}$ is then the smallest admissible set $\mathscr{A}$ inside V_A such that the structure $\mathfrak{A}$ is $\mathscr{A}$-finite (the notion of admissible set is a straightforward modification of the urelement-free notion; for details, cf. *loc. cit.*).

The *pure* sets in V_A are those whose transitive closure does not contain any urelement (i.e., any element of A).

A simple result connecting $\mathrm{HYP}_{\mathfrak{A}}$ and a model theoretical property of $\mathfrak{A}$ that is similar to what we will prove below appeared in [15] and is due to the second author of the present paper. It says that if $\mathfrak{A}$ is countable, then for having a *pure* $\mathfrak{A}' \in \mathrm{HYP}_{\mathfrak{A}}$ (with pure domain A') and an isomorphism $f\colon \mathfrak{A}' \simeq \mathfrak{A}$ *in* $\mathrm{HYP}_{\mathfrak{A}}$ it is necessary and sufficient that there is an expansion $(\mathfrak{A}, a_1, \cdots, a_n)$ of $\mathfrak{A}$ with finitely many elements which is rigid. This result describes when it is true that the pure part of $\mathrm{HYP}_{\mathfrak{A}}$ 'fixes' (or 'describes') $\mathfrak{A}$ in a strongest possible way. Below we will encounter a weakening of this situation.

DEFINITION 4.1. Let $\mathfrak{A}$ be a structure of a finite similarity type and $\mathcal{A} = \mathrm{HYP}_{\mathfrak{A}}$. Among the $\mathcal{A}$-finite subsets of A we single out the *pseudo-pure* (p.p.) ones according to the following inductive definition. The collection of $\mathcal{A}$-finite p.p. subsets of A form the smallest set P such that

(1) For every $a \in A$, we have $\{a\} \in P$.

(2) Any $\mathcal{A}$-finite subset of an element of P belongs to P.

(3) If I is a *pure* set in $\mathcal{A}$ and $\langle A_i\colon i \in I\rangle$ is an $\mathcal{A}$-finite family of sets $A_i \subset A$ each belonging to P, then $\bigcup_{i\in I} A_i \in P$.

In formulating the above mentioned and the following theorem, we were helped by suggestions of Leo Harrington.

THEOREM 4.2. *Let $\mathfrak{A}$ be a countable structure of finite similarity type. Then the following are equivalent*:

(i) *$\mathfrak{A}$ is the model of a categorical $L_{\omega_1\omega}$-sentence, i.e., the Scott sentence of $\mathfrak{A}$ is categorical.*

(ii) *A is pseudo-pure in $\mathcal{A} = \mathrm{HYP}_{\mathfrak{A}}$.*

PROOF. First we consider the easy direction (ii) $\Rightarrow$ (i). Notice that we have:

(4.3) Every p.p. $X \subset A$ is of cardinality $\leqq \operatorname{card} \mathcal{A}_{\mathrm{pp}}$ where $\mathcal{A}_{\mathrm{pp}}$ is the set of pure sets in $\mathcal{A}$ (the *pure part* of $\mathcal{A}$).

This is proved by an easy induction on the clauses of Definition 4.1.

Secondly, recall that

(4.4) If $\mathfrak{B} \equiv_{\infty\omega} \mathfrak{A}$, then $(\mathrm{HYP}_{\mathfrak{B}})_{\mathrm{pp}} = (\mathrm{HYP}_{\mathfrak{A}})_{\mathrm{pp}}$ (cf. e.g. [15]).

Assume that $\mathfrak{A}$ is countable, A is p.p. in $\mathrm{HYP}_{\mathfrak{A}}$ and $\mathfrak{B} \equiv_{\infty\omega} \mathfrak{A}$. Then by (4.3) and (4.4), $\operatorname{card} B \leqq \operatorname{card} (\mathrm{HYP}_{\mathfrak{B}})_{\mathrm{pp}} = \operatorname{card} (\mathrm{HYP}_{\mathfrak{A}})_{\mathrm{pp}} = \aleph_0$. Hence $\mathfrak{B} \simeq \mathfrak{A}$ as required.

Turning to the other direction (i) $\Rightarrow$ (ii), we first invoke a result due to Nadel [14].

THEOREM 4.5. [14]. *Let $\mathscr{A}$ be a countable admissible set, possibly with urelements and $\mathfrak{A}$ an $\mathscr{A}$-finite structure (in the application, $\mathscr{A} = \mathrm{HYP}_{\mathfrak{A}}$). If there is an uncountable $\mathfrak{B}$ such that $\mathfrak{A} \equiv_{L_{\mathscr{A}}} \mathfrak{B}$ then there is an uncountable $\mathfrak{B}'$ such that $\mathscr{A} \equiv_{\infty\omega} \mathfrak{B}'$.*

For completeness, we sketch a proof of Theorem 4.5 using $\mathscr{A}$-r.s. models as well as two additional facts:

(4.6) (Nadel's basis theorem; cf. [13] or [1]). If $\mathfrak{A}, \mathfrak{A}_1$ are $\mathscr{B}$-finite structures for some admissible $\mathscr{B}$ and $\mathfrak{A} \equiv_{L_{\mathscr{B}}} \mathfrak{A}_1$ then $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{A}_1$.

(4.7) If $\mathfrak{A}$ is $\mathscr{A}$-finite then for every $\beta < \alpha = \mathrm{Ord}_{\mathscr{A}}$ and $\boldsymbol{a} \in A$ there is a formula $\phi^{\boldsymbol{a}}_{\beta} \in L_{\mathscr{A}}$ such that for all $\mathfrak{A}_1$, $\mathfrak{A}_1 \models \phi^{\boldsymbol{a}}_{\beta}[\boldsymbol{a}_1]$ iff $(\mathfrak{A}, \boldsymbol{a}) \equiv^{\beta}_{\infty\omega} (\mathfrak{A}_1, \boldsymbol{a}_1)$ ($\phi^{\boldsymbol{a}}_{\beta}$ is the "canonical β-type" of $\boldsymbol{a}$, cf. [1], page 298).

Suppose $\mathfrak{A}$ and $\mathscr{A}$ are as in Theorem 4.5. By (4.6), $\mathfrak{A}$ is $L_{\mathscr{A}}$-homogeneous. $T = \mathrm{Th}_{L_{\mathscr{A}}}(\mathfrak{A})$ is a $\Sigma_{\mathscr{A}}$-theory and $\mathfrak{A}$, being $\mathscr{A}$-finite, is an $\mathscr{A}$-r.s. model of it; it is the *only* $\mathscr{A}$-r.s. model of T. (Indeed, if $\mathfrak{A}_1$ is any other such then by Theorem 2.2 (ii), $\mathfrak{A}_1$ is $\mathscr{B}$-finite for some countable admissible $\mathscr{B} \supset \mathscr{A}$ with $\mathrm{Ord}_{\mathscr{B}} = \mathrm{Ord}_{\mathscr{A}} = \alpha$; as $\mathfrak{A}_1 \models T$, we have that $\mathfrak{A}_1 \equiv_{L_{\mathscr{A}}} \mathfrak{A}$ and, by (4.7), $\mathfrak{A}_1 \equiv_{L_{\mathscr{B}}} \mathfrak{A}$; by (4.6), $\mathfrak{A}_1 \equiv_{\infty\omega} \mathfrak{A}$ hence by the countability of $\mathfrak{A}_1$, $\mathfrak{A}_1 \cong \mathfrak{A}$.)

Since T has an uncountable model and T is an $L_{\mathscr{A}}$-complete $\Sigma_{\mathscr{A}}$-theory, every $\mathscr{A}$-r.s. model of T has a proper $L_{\mathscr{A}}$-elementary extension which is again $\mathscr{A}$-r.s. by Theorem 2.2 (i). $\mathfrak{A}$ is $\mathscr{A}$-r.s. Now it follows that we can build an ω_1 chain of models $\mathfrak{A}_\alpha$ ($\alpha < \omega_1$) such that $\mathfrak{A}_\alpha \simeq \mathfrak{A}$, $\mathfrak{A}_\alpha \prec_{L_{\mathscr{A}}} \mathfrak{A}_{\alpha+1}$ and $\mathfrak{A}_{\lambda_i} = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha$ for limit $\lambda < \omega_1$. For $\mathfrak{A}' = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$, one sees that $\mathfrak{A}' \equiv_{\infty\omega} \mathfrak{A}$ by showing that $L_{\mathscr{A}}$-equivalence of finite tuples in $\mathfrak{A}'$ and $\mathfrak{A}$ is a partial isomorphism (or: back and forth property) between $\mathfrak{A}'$ and $\mathfrak{A}$ (this argument uses the $L_{\mathscr{A}}$-homogeneity of $\mathfrak{A}$).

This completes our sketch for the proof of Nadel's Theorem 4.5. We now return to the proof of Theorem 4.2.

Assume (i) in Theorem 4.2. Then by Theorem 4.5, $\mathfrak{B} \equiv_{L_{\mathscr{A}}} \mathfrak{A}$ implies that $\mathfrak{B} \simeq \mathfrak{A}$. Let $T = \mathrm{Th}_{L_{\mathscr{A}}}(\mathfrak{A})$, the $L_{\mathscr{A}}$-theory of $\mathfrak{A}$. Notice that, having construed L as a *pure* (finite) set as we can assume without loss of generality, we have $L_{\mathscr{A}} = L_{\mathscr{A}_{\mathrm{pp}}}$. We can apply Gregory's Theorem 0.1 to this T and we conclude that there is $\theta \in \mathscr{C}_{L_{\mathscr{A}}}(v = v)$ such that $T \models \theta$, i.e., $\mathfrak{A} \models \theta$.

CLAIM 4.8. *If* $\theta(y) \in \mathscr{C}(\phi(x, y))$ *and* $\mathfrak{A} \models \theta[\boldsymbol{b}]$, *then*

$$\phi(\mathfrak{A}, \boldsymbol{b})(=\{a \in A : \mathfrak{A} \models \phi(a, \boldsymbol{b})\}) \text{ is p.p.}$$

The *proof* of Claim 4.8 is by an induction paralleling the definition of $(\cdot) \in \mathscr{C}(\cdot)$. We leave the straightforward details to the reader.

Returning to the proof of the theorem, we have $\theta \in \mathscr{C}_{L_{\mathscr{A}}}(v = v)$ such that $\mathfrak{A} \models \theta$. But by Claim 4.8 this means that $A(=\{a \in A : \mathfrak{A} \models a = a\})$ is p.p. Q.E.D.

§5. Stationary logic

Logic with the generalized quantifiers "for almost all countable subsets s" was introduced by S. Shelah and studied in [2]; cf. the references there. Here we discuss a variant of the logic considered in [2]. As a consequence of our completeness theorem for this variant, we obtain a new proof of the $\mathscr{A}$-recursive enumerability of validities in $L_{\mathscr{A}}(\text{aa})$, for any countable admissible $\mathscr{A}$ and for $L_{\mathscr{A}}(\text{aa})$ as defined in [2], as well as Σ-compactness for this logic, but we do not obtain the precise axiomatization given in [2]. Nevertheless, our axiomatization is equally specific and intuitive; also it gives rise to 'Gregory type' conclusions (cf. Theorem 0.1 in the Introduction). The differences between our completeness result and that in [2] are quite similar to those between the Completeness Theorem 1.2 and Keisler's [9]. A minor difference will be that instead of the aa quantifier, we use stat as primitive.

Let L be a (possibly) multisorted language and let R be a binary relation symbol with first argument of sort U and second argument of sort V; U and V might be identical. We shall use the letters "u" and "v" with or without subscripts to denote variables of sorts U and V respectively; also, "x" and "y" with or without subscripts, will denote arbitrary variables. We introduce a quantifier, 'stat' acting on variables of sort V and define the language $L_{\omega_1\omega}(\text{stat})$ in the obvious way, allowing the formation rule: if $\phi \in L_{\omega_1\omega}(\text{stat})$, then so is $\text{stat}\, v\, \phi$ (read: for stationary many v, $\phi(v)$). We write $\text{aa}v\phi$ as an abbreviation for $\neg\, \text{stat}\, v \neg\, \phi$. Read $\text{aa}v\phi$ as 'for almost all v, ϕ'. We can speak of $L_{\mathscr{A}}(\text{stat})$, $\mathscr{A}$ countable admissible and define fragments F of $L_{\mathscr{A}}(\text{stat})$ precisely as we did in §3 for $L_{\mathscr{A}}(Q)$ (stat playing the role of Q). Notice that F is closed under applications of the first order quantifiers $\forall$, $\exists$ to variables of every sort, including V. Also, considering each $\text{stat}\ v\ \phi(v, \boldsymbol{x}) \in F$ as a new atomic formula we can speak of (weak) F-structures. Recall (cf. [2] and references there) that a subset S of $\mathscr{P}_{\omega_1}(A)$ (= the set of countable subset of A) is called *closed and unbounded* (c.u.b.) if S is *closed* under unions of countable increasing sequences, and every countable subset of A is contained in some member of S. S is called *stationary* if its complement $\mathscr{P}_{\omega_1}(A) - S$ contains no c.u.b. set. The semantics of $L_{\omega_1\omega}(\text{stat})$ is described by:

DEFINITION 5.1. An F-structure $\mathfrak{A}$ is called a standard F-structure if

(i) for every $b \in V$, $R(\mathfrak{A}, b) = \{a : a \in U^{\mathfrak{A}}$ and $\mathfrak{A} \models R[a, b]\}$ is countable;

(ii) if stat $v\ \phi \in F$ then

$\models$ stat $v\ \phi[v, \boldsymbol{c}]$ iff the set $S_{\phi,c} = \{R(\mathfrak{A}, b) : b \in V^{\mathfrak{A}}$ and $\mathfrak{A} \models \phi[b, \boldsymbol{c}]\}$ is a stationary subset of $\mathscr{P}_{\omega_1}(U^{\mathfrak{A}})$;

(iii) the set $\mathscr{P}^{\mathfrak{A}} = \{R(\mathfrak{A}, b) : b \in V^{\mathfrak{A}}\}$ contains a c.u.b. subset of $\mathscr{P}_{\omega_1}(U^{\mathfrak{A}})$.

REMARK. While condition (ii) insures that "stationary" has the standard meaning in $\mathfrak{A}$, condition (iii) comes to insure that "non-stationary" and, hence, "almost all" have the standard meaning. Indeed, (iii) implies the following two conditions:

(iii)′ if stat $v\phi \in F$ then $\mathfrak{A} \models \neg$stat $v\phi[v, \boldsymbol{c}]$ iff $\mathscr{P}^{\mathfrak{A}} - S_{\phi,c} = \{R(\mathfrak{A}, b) : b \in V^{\mathfrak{A}}$ and $\mathfrak{A} \not\models \phi[b, \boldsymbol{c}]\}$ contains a c.u.b. subset of $\mathscr{P}_{\omega_1}(U^{\mathfrak{A}})$.

(iii)″ if aa$v\phi \in F$ then $\mathfrak{A} \models$ aa$v\phi[v, \boldsymbol{c}]$ iff $S_{\phi},\boldsymbol{c}$ contains a c.u.b. subset of $\mathscr{P}_{\omega_1}(U^{\mathfrak{A}})$.

While the implications (iii)→(iii)′→(iii)″ are trivially true, the converse implications do not hold unless F is rich enough (e.g. if stat $v\ v \neq v \in F$ then (iii)″→(iii)).

A *valid* sentence of F is one which is true in all standard F-structure.. If $T \cup \{\phi\} \subset F$ then $T \models \phi$ iff ϕ is true in every standard model of T.

EXAMPLE 1. Assume that the sorts U and V are identical and let us restrict ourselves to structures in which R, denoted in this context by "$<$", is a linear ordering of U. It is easy to see that a standard F-structure will be one in which:

(a) $(U, <)$ is a *strongly* $\aleph_1$*-like* ordering, i.e. an $\aleph_1$-like ordering into which ω_1 can be continuously and confinally embedded.

(b) stat $u\ \phi(u)$ means that $\phi(u)$ is satisfied by a stationary set of elements of U (equivalently, every set of elements of U which is a continuous image of ω_1 contains elements satisfying U).

EXAMPLE 2. Assume that L contains only the distinct sorts U and V. Let us restrict ourselves to those L-structures $\mathfrak{A}^*$ whose U-sort is a set A while the V-sort is a collection of countable subsets of A containing a c.u.b. set and such that $R^{\mathfrak{A}^*}$ is the membership relation $\in$. In this context, we usually denote R by $\in$, the variables of sort U by x_i, y_i and those of sort V by s_i, $i < \omega$. Let L_0 be the set of those L-symbols whose arguments are all of sort U (thus, $\in$ is *not* in L_0). With each $\mathfrak{A}^*$ we naturally associate the L_0-structure $\mathfrak{A}$ whose universe is A and which satisfies that $P^{\mathfrak{A}} = P^{\mathfrak{A}^*}$ for all P in L_0. The same L_0-structure $\mathfrak{A}$ is, of course, associated with many L-structures $\mathfrak{A}^*$; the same $L_{\omega_1\omega}(\text{stat})$ sentence ϕ may be true in one of these and false in another. Let $L^-_{\omega_1\omega}(\text{stat})$ be the set of those $L_{\omega_1\omega}(\text{stat})$ sentences which do not contain quantifiers $\forall s$ or $\exists s$ with s of sort V. If $L = L_0(\in)$ then, in contrast to what has been said before, the truth value of a sentence $\phi \in L^-_{\omega_1\omega}(\text{stat})$ in $\mathfrak{A}^*$ depends only on $\mathfrak{A}$; we write "$\mathfrak{A} \models \phi$" instead of "$\mathfrak{A}^* \models \phi$" in this situation. The logic $L^-_{\omega_1\omega}(\text{stat})$ has been studied in [2].

We now return to the general situation and associate with each formula $\phi(v, \boldsymbol{x})$ in a given fragment F (of $L_{\mathscr{A}}(\text{stat})$) a set of formulas $\mathscr{C}_F(\phi)$ with free variables $\boldsymbol{x}$.

We use the abbreviation $v \subset v'$ for $\forall u[Ruv \rightarrow Ruv']$.

DEFINITION 5.2. The sets $\mathscr{C}_F(\phi)$ for $\phi = \phi(v, \boldsymbol{x}) \in F$ are defined by the following clauses subject to the restriction that all formulas mentioned are in F.

(1a) $\mathrm{t}(=\text{true}) \in \mathscr{C}(\neg\,\text{stat}\, v'\, \phi(v') \wedge \phi(v))$.

(1b) $\mathrm{t} \in \mathscr{C}(\neg\, Ruv)$.

(1c) $\mathrm{t} \in \mathscr{C}(\neg\, v' \subset v)$.

(2)

(3) as in Definition 0.2.

(p.c.)

(4) if $\theta \in \mathscr{C}(\phi)$, then $\forall u\theta \in \mathscr{C}(\exists u(Ruv \wedge \phi))$.

The rationale behind $\mathscr{C}$ is contained in

LEMMA 5.3. *Every standard structure is a model of the universal closure of*

$$\theta \rightarrow \neg\,\text{stat}\, v\,\phi(v)$$

whenever $\theta \in \mathscr{C}(\phi)$.

The *proof* is by induction according to the clauses of Definition 5.2; e.g., clause (2), the union clause, will correspond to the fact that a countable union of non-stationary sets is non-stationary. To deal with clause (4), assume that $\theta \in \mathscr{C}_F(\phi)$ and (by the induction hypothesis) that $\theta(u) \rightarrow \text{aa}v \neg\, \phi(u, v)$ is true, where we have switched from using stat v to aa$v$. To have $\forall u\theta(u) \rightarrow \text{aa}v \neg \exists u(Ruv \wedge \phi(u, v))$ as required, hence it is sufficient to have $\forall u\ \text{aa}v \neg\, \phi(u, v) \rightarrow \text{aa}v \forall u(Ruv \rightarrow \neg\, \phi(u, v))$. This is the Kueker diagonalization principle, which is the main axiom in [2]. We note that it is closely related to the Fodor regressive function theorem, cf. [2] and references there.

DEFINITION 5.4. The set of F-axioms, $T_0(F)$, consists of the universal closures of the formulas $\theta \to \neg \operatorname{stat} v\, \phi(v)$ for $\theta \in \mathscr{C}_F(\phi)$ and $\operatorname{stat} v\, \phi(v) \in F$ as well as $\neg \theta$ for $\theta \in \mathscr{C}_F(v = v)$.

The F-axioms are valid in all standard models. This follows from Lemma 5.3 as well as from the fact that $\operatorname{stat} v(v = v)$ is true in the standard models.

We can now state a completeness theorem for $L_{\mathscr{A}}(\operatorname{stat})$ in forms similar to either Theorem 1.2 or Theorem 1.3. For example, the analogue of Theorem 1.3 is:

THEOREM 5.5. *Let T be a $\Sigma_{\mathscr{A}}$-subset of an admissible fragment $F \subset L$ (stat). If $T \cup T_0(F)$ has a weak model then it has a standard model.*

The *proof* of Theorem 5.5 parallels the proof of Theorem 1.3; therefore we mention only the main points. It turns out that the game sentence needed for Theorem 5.5 is somewhat simpler than that for Theorem 1.3.

Keisler's method for building the standard model is replaced by a similar one used in [2]. For countable F-structures $\mathfrak{A}$ and $\mathfrak{B}$ we say that $\mathfrak{B}$ is a *blunt* extension of $\mathfrak{A}$ (*via* b_0) if:

(a) $\mathfrak{A} \prec \mathfrak{B}$ (again, "$\prec$" means F-elementary inclusion),

(b) $b_0 \in V^{\mathfrak{B}}$ and $U^{\mathfrak{A}} = R(\mathfrak{B}, b_0) = \{a : a \in U^{\mathfrak{B}} \text{ and } \mathfrak{B} \models R[a, b_0]\}$,

(c) for $b \in V^{\mathfrak{A}}$, $R(\mathfrak{A}, b) = R(\mathfrak{B}, b)$,

(d) for any $\operatorname{stat} v\, \phi \in F$, if $\mathfrak{A} \models \neg \operatorname{stat} v\, \phi[v, \boldsymbol{c}]$ then $\mathfrak{B} \models \neg \phi[b_0, \boldsymbol{c}]$.

Given a $\Sigma_{\mathscr{A}}$-theory T, assume that we have an F-elementary chain of countable structures $\{\mathfrak{A}_\alpha\}_{\alpha<\omega}$ such that

(1) $\mathfrak{A}_\alpha \models T$,

(2) $\mathfrak{A}_{\alpha+1}$ is a blunt extension of $\mathfrak{A}_\alpha$, *via* b_α,

(3) for each $\operatorname{stat} v\, \phi(v, \boldsymbol{x}) \in F$ and $\boldsymbol{c} \in A_\alpha$, there is a *stationary* set $X = X_{\operatorname{stat} v\phi, \boldsymbol{c}} \subset \omega_1 - \alpha$ such that if $\mathfrak{A}_\alpha \models \operatorname{stat} v\, \phi[v, \boldsymbol{c}]$ then for each $\beta \in X$, $\mathfrak{A}_{\beta+1} \models \phi[b_\beta, \boldsymbol{c}]$.

Under these assumptions, $\mathfrak{A}_{\omega_1} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$ is a standard model of T.

Assuming that $T \cup T_0(F)$ has a model we want to build a chain as above with each $\mathfrak{A}_\alpha$ and $\mathscr{A}$-r.s. model. As in the proof of Theorem 1.3 we see (using the fact that ω_1 has uncountably many mutually disjoint stationary subsets) that all we have to prove is the following.

LEMMA 5.6. *If $\mathfrak{A}$ is an $\mathscr{A}$-r.s. model of $T_0(F)$ then for each $\boldsymbol{c} \in A$ there is a substructure $\mathfrak{A}_0 \prec \mathfrak{A}$ and an element $b_0 \in V^{\mathfrak{A}}$ such that $\boldsymbol{c} \in A_0$ and $\mathfrak{A}$ is a blunt extension of $\mathfrak{A}_0$ via b_0; moreover, if* $\operatorname{stat} v\, \phi(v, \boldsymbol{x}) \in F$ *and* $\mathfrak{A} \models \operatorname{stat} v\, \phi[v, \boldsymbol{c}]$ *then b_0 can be chosen so that $\mathfrak{A} \models \phi[b_0, \boldsymbol{c}]$.*

The *proof* of Lemma 5.6 parallels that of Theorem 1.3. We let $\Phi(v, \boldsymbol{x})$ be a game sentence expressing that:

> "There is a substructure $\mathfrak{A}_0 < \mathfrak{A}$ such that $\boldsymbol{x} \in A_0$ and $\mathfrak{A}$ is a blunt extension of $\mathfrak{A}_0$ via v".

We have to show:

(4) $\mathfrak{A} \models \exists v \Phi(v, \boldsymbol{x})$

and

(5) $\mathfrak{A} \models \text{stat } v\ \phi \rightarrow \exists v(\phi(v, \boldsymbol{x}) \wedge \Phi(v, \boldsymbol{x})$ for all stat $v\ \phi \in F$.

By the method of proof of Theorem 1.3, clause (4) is shown to follow from the fact that $\mathfrak{A} \models \neg\theta$ for all $\theta \in \mathscr{C}(v = v)$ and clause (5) follows from the fact that $\mathfrak{A} \models \text{stat } v\ \phi \rightarrow \neg\theta$ whenever $\theta \in \mathscr{C}(\phi)$.

We end the (sketch of the) proof by explicitly writing down $\Phi(v, \boldsymbol{x})$ and leaving all the other details to the reader.

Two notational tricks will simplify this final task. Let $y_0, y_1, \cdots$ be variables of a special sort which range over the *whole* universe; let $\gamma(y_i, v)$ mean Ry_iv whenever the value of y_i is of sort U, $y_i \subset v$ whenever this value is of sort V and t otherwise. Finally, let ρ be the conjunction of all formulas of the form Ruv, $v' \subset v$ where u and v' appear in $\boldsymbol{x}$ (note that the variable v is fixed). Then we put

$$\Phi(v, \boldsymbol{x}) = \forall u_0 \bigwedge_{\delta_0} \exists y_0 \bigwedge_{\beta_0} \forall u_1 \cdots \left(\bigwedge_{n<\omega} (\delta_n \wedge \beta_n \wedge \gamma(y_n, v) \wedge \rho) \right)$$

where:

(i) u_n is restricted to Ru_nv,

(ii) δ_n ranges over all F-formulas of the form

$$\exists y \chi(\boldsymbol{x}, \boldsymbol{u}_n, \boldsymbol{y}_{n-1}, y) \rightarrow \chi(\boldsymbol{x}, \boldsymbol{u}_n, \boldsymbol{y}_n),$$

(iii) β_n ranges over all F-formulas of the form

$$\neg \text{stat } v' \phi(v', \boldsymbol{x}, \boldsymbol{u}_n, \boldsymbol{y}_n) \rightarrow \neg \phi(v, \boldsymbol{x}, \boldsymbol{u}_n, \boldsymbol{y}_n).$$

The matrix of Ψ (i.e., $\Phi\neg$) should be written as

$$\bigvee_{n<\omega} (\delta_n \rightarrow \neg \beta_n \vee \neg \gamma(y_n, v) \vee \neg \rho).$$

We conclude with an application and a few remarks.

Let us situate ourselves in the context of Example 1 and ask ourselves when does a $\boldsymbol{\Sigma}_{\mathscr{A}}$-theory $T \subset L_{\mathscr{A}}$ have a model with $(U, <)$ a strongly $\aleph_1$-like ordering. Answer: this happens iff T has an F-standard model iff T is consistent with $T_0(F)$ where $F = L_{\mathscr{A}}$ (notice that $T_0(F)$ is nonvoid since it contains $\neg\theta$ for all

$\theta \in \mathscr{C}(v = v)$!). To state this result in the style of Theorem 0.1 one has to define $\mathscr{D}(\phi)$ for every $\phi(v, \mathbf{x}) \in L_{\mathscr{A}}$ precisely as in Definition 5.2 except that clauses (1a), (1b), (1c) are simplified to the single clause:

(1) t (= true) $\in \mathscr{D}(v \leqq u)$.

THEOREM 5.7. *For a $\Sigma_{\mathscr{A}}$-theory $T \subset L_{\mathscr{A}}$, the following are equivalent*:

(i) *T has no model in which $(U, <)$ is a strongly $\aleph_1$-like ordering.*

(ii) *T has no countable models $\mathfrak{A}_0, \mathfrak{A}_1$ such that $\mathfrak{A}_1$ is a blunt extension of $\mathfrak{A}_0$.*

(iii) *$T \vdash \theta$ for some $\theta \in \mathscr{D}(v = v)$.*

((ii) implies (i) because (ii) implies $\mathfrak{A}_0 \models T_0(F)$ with $F = L_{\mathscr{A}}$.)

REMARK. We refer back to the remark following Definition 5.1. By substituting (iii)′ or (iii)″ into (iii) or by dropping (iii) in that definition, we obtain slightly different logics. To get sound and complete systems for these we have to alter $T_0(F)$ accordingly. Thus: If (iii)′ (or (iii)″) replaces (iii) in Definition 5.1 then the axioms $\neg\theta$, $\theta \in \mathscr{C}(v = v)$ should be replaced by

$$\{\neg \operatorname{stat} v\phi \to \neg\theta \colon \theta \in \mathscr{C}(v = v), \operatorname{stat} v\phi \in F\}$$

(respectively by $\{\operatorname{aa} v\,\phi \to \neg\theta \colon \theta \in \mathscr{C}(v = v), \operatorname{aa} v \in F\}$) in Definition 5.4. If (iii) is dropped altogether from Definition 5.1 then the axioms $\neg\theta$, $\theta \in \mathscr{C}(v = v)$ should be dropped from Definition 5.4.

REFERENCES

1. J. Barwise, *Admissible Sets and Structures*, Springer–Verlag, 1975.

2. J. Barwise, M. Kaufmann and M. Makkai, *Stationary logic*, Ann. Math. Logic, to appear.

3. J. Barwise and J. Schlipf, *An introduction to recursively saturated and resplendent models*, J. Symbolic Logic **41** (1976), 531–536.

4. Kim B. Bruce, *Model-theoretic forcing in logic with a generalized quantifier*, Ann. Math. Logic, to appear.

5. J. Gregory, *Uncountable models and infinitary elementary extensions*, J. Symbolic Logic **38** (1973), 460–470.

6. V. Harnik, *Refinements of Vaught's normal form theorem*, J. Symbolic Logic, to appear.

7. V. Harnik and M. Makkai, *Applications of Vaught sentences and the covering theorem*, J. Symbolic Logic **41** (1976), 171–187.

8. H. J. Keisler, *Some model-theoretic results for ω-logic*, Israel J. Math. **4** (1966), 249–261.

9. H. J. Keisler, *Logic with the quantifier "there exist uncountably many"*, Ann. Math. Logic **1** (1970), 1–93.

10. H. J. Keisler, *Model Theory for Infinitary Logic*, North-Holland, 1971.

11. K. Kunen, *Combinatorics*, in *Handbook of Mathematical Logic* (J. Barwise ed.), North-Holland, 1977, 371–401.

12. M. Makkai, *Admissible sets and infinitary logic*, in *Handbook of Mathematical Logic* (J. Barwise, ed.), North-Holland, 1977, 233–281.

13. M. Nadel, *Scott sentences and admissible sets*, Ann. Math. Logic **7** (1974), 267–294.

14. M. Nadel, *On models $\equiv_{\infty\omega}$ to an uncountable model*, Proc. Amer. Math. Soc. **54** (1976), 307.

15. M. Nadel and J. Stavi, *The pure part of* HYP(M), J. Symbolic Logic **42** (1977), 33–46.

16. J. P. Ressayre, *Boolean models and infinitary first order languages*, Ann. Math. Logic **6** (1973), 41–92.

17. J. P. Ressayre, *Models with compactness properties relative to an admissible language*, Ann. Math. Logic **11** (1977), 31–55.

18. S. Shelah, *Generalized quantifiers and compact logic*, Trans. Amer. Math. Soc. **204** (1975), 342–364.

19. R. Vaught, *Descriptive set theory in* $L_{\omega_1\omega}$ in *Cambridge Summer School in Mathematical Logic*, Lecture Notes in Math. No. 337, Springer–Verlag, 1973, pp. 574–598.

UNIVERSITY OF HAIFA,
HAIFA, ISRAEL

AND

MCGILL UNIVERSITY
MONTREAL P.Q. H3A 2K6 CANADA